HW2 SOLUTIONS

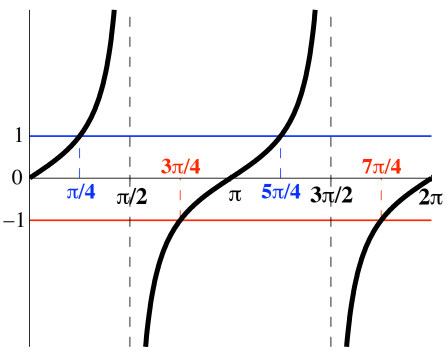
MAT 1320D Winter 2009

Problem 1.

(a) $|\tan x| = 1$

$\iff \tan x = -1$ or $\tan x = 1$

$\iff x = \frac{3\pi}{4}, \frac{7\pi}{4}$ or $x = \frac{\pi}{4}, \frac{5\pi}{4}$



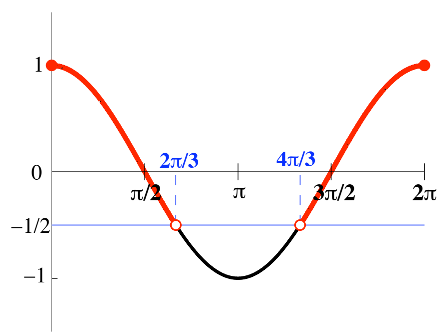
(b) $2\cos x + 1 > 0$

$\Rightarrow 2\cos x > -1$

$\Rightarrow \cos x > -1/2.$

$\cos x = -\frac{1}{2}$ when $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

The figure below shows that $\cos x > -\frac{1}{2}$ when $0 \le x < \frac{2\pi}{3}$, $\frac{4\pi}{3} < x \le 2\pi$.



Problem 2.

(a) $$\lim_{h\to 0} \frac{(3+h)^{-1} - 3^{-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h} = \lim_{h\to 0} \frac{-h}{3h(3+h)}$$

$$= \lim_{h\to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}.$$

(b) $-1 \le \sin x(\pi/x) \le 1$

$\Rightarrow e^{-1} \le e^{\sin(\pi/x)} \le e^{1}$

$\Rightarrow \frac{\sqrt{x}}{e} \le \sqrt{x} e^{\sin(\pi/x)} \le \sqrt{x}\, e.$

Since $\lim_{x\to 0^+} \frac{\sqrt{x}}{e} = 0$ and $\lim_{x\to 0^+} \sqrt{x}\, e = 0$,

$$\lim_{x\to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0$$

by the Squeeze Theorem.

(c) $$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \frac{x+2}{\sqrt{x^2\left(9+\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{x+2}{\sqrt{x^2}\sqrt{9+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{x+2}{|x|\sqrt{9+\frac{1}{x^2}}}$$

($|x| = x \ \because x > 0$ as $x \to \infty$)

$$= \lim_{x\to\infty} \frac{x+2}{x\sqrt{9+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}} = \frac{1}{3}$$

Problem 3.

By Theorem 5 (page 120 in the textbook), each piece of F is continuous on its domain. We need to check for continuity at $r = R$.

- $F(R) = \frac{GM}{R^2}$
- $\lim_{r\to R^-} F(r) = \lim_{r\to R^-} \frac{GMr}{R^3} = \frac{GM}{R^2}$

  and $\lim_{r\to R^+} F(r) = \lim_{r\to R^+} \frac{GM}{r^2} = \frac{GM}{R^2}$.

  $\therefore \lim_{r\to R} F(r) = \frac{GM}{R^2}$.

Since $F(R) = \frac{GM}{R^2}$ *and* $\lim_{r\to R} F(r) = \frac{GM}{R^2}$,

- $\lim_{r\to R} F(r) = F(R)$.

Hence, F is continuous at $r = R$.

Therefore, F is a continuous function of r.

Problem 4.

$g(x) = 1 - x^3$

$$g'(0) = \lim_{\Delta x\to 0} \frac{g(0+\Delta x) - g(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\left[1-(0+\Delta x)^3\right] - (1-0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\left[1-(\Delta x)^3\right] - 1}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \left[-(\Delta x)^2\right] = 0$$

♣ $g'(0)$ can also be obtained by the following way:

$$g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{1-x^3-(1-0)}{x} = \lim_{x\to 0}\left(-x^2\right) = 0$$

Thus, an equation of the tangent line is

$$y-1 = 0(x-0) \quad \text{or} \quad y = 1.$$

Problem 5.

$s = t^2 - 8t + 18$ where t is measured in seconds.

(a) The average velocity over each time interval

(i) $[3, 4]$

$$\frac{\Delta s}{\Delta t} = \frac{s(4)-s(3)}{4-3} = \frac{(4^2-8\cdot 4+18)-(3^2-8\cdot 3+18)}{1} = -1 \text{ m/s}.$$

(ii) $[3.5, 4]$

$$\frac{\Delta s}{\Delta t} = \frac{s(4)-s(3.5)}{4-3.5} = \frac{(4^2-8\cdot 4+18)-(3.5^2-8\cdot 3.5+18)}{0.5} = -0.5 \text{ m/s}.$$

(iii) $[4, 5]$

$$\frac{\Delta s}{\Delta t} = \frac{s(5)-s(4)}{5-4} = \frac{(5^2-8\cdot 5+18)-(4^2-8\cdot 4+18)}{1} = 1 \text{ m/s}.$$

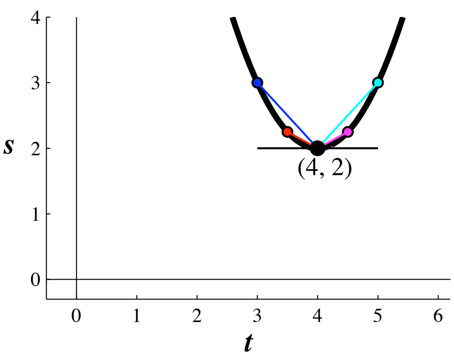
(iii) $[4, 4.5]$

$$\frac{\Delta s}{\Delta t} = \frac{s(4.5)-s(4)}{4.5-4} = \frac{(4.5^2-8\cdot 4.5+18)-(4^2-8\cdot 4+18)}{0.5} = 0.5 \text{ m/s}.$$

(b) The instantaneous velocity when $t = 4$

$$\lim_{t\to 4} \frac{s(t)-s(4)}{t-4} = \lim_{t\to 4} \frac{(t^2-8t+18)-(4^2-8\cdot 4+18)}{t-4} = \lim_{t\to 4} \frac{t^2-8t+16}{t-4} = \lim_{t\to 4} \frac{(t-4)^2}{t-4} = \lim_{t\to 4}(t-4) = 0.$$

(c)



Problem 6.